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Exact solutions of the multidimensional derivative nonlinear Schrödinger equation for many-body systems near criticality

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Abstract. We investigate the problem of strongly interacting many-body systems near criticality as recently described in terms of nonlinear dynamics by Dixon and Tuszyński. We show that in the first order of approximation, the equation of motion for the order parameter can be mapped on the derivative nonlinear Schrödinger equation and thus can be solved exactly. We solve the equations both in the absence and presence of current densities.

1. Physical introduction

In a series of two recent papers (Tuszyński and Dixon 1989, Dixon and Tuszyński 1989) a new method has been presented to treat strongly interacting many-body systems described using the generic effective Hamiltonian in the form

$$H_{\text{eff}} = \sum_{\boldsymbol{k},\boldsymbol{l}} \omega_{\boldsymbol{k},\boldsymbol{l}} q_{\boldsymbol{k}}^{\dagger} q_{\boldsymbol{l}} + \sum_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} \Delta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} q_{\boldsymbol{k}}^{\dagger} q_{\boldsymbol{l}} q_{\boldsymbol{m}}^{\dagger} q_{\boldsymbol{k}+\boldsymbol{l}-\boldsymbol{m}} \tag{1.1}$$

where $q_{\boldsymbol{k}}^{\dagger}$ and $q_{\boldsymbol{l}}$ are creation and annihilation operators with $\boldsymbol{k}$ and $\boldsymbol{l}$, respectively. A large number of important physical systems can be described using this type of Hamiltonian, e.g., conduction electrons in a metal, superconductors, superfluids, anharmonic crystal lattices, ferromagnets, etc. This approach applies to both fermions and bosons. The method proposed consists of expanding the interaction coefficients ω and Δ in a Taylor series in powers of momentum components with respect to a critical point $(\boldsymbol{\eta}_0, \boldsymbol{k}_0, \boldsymbol{m}_0)$ in momentum space. A quantum field operator is defined as

$$\psi(\boldsymbol{r}) = \Omega^{-1/2} \sum_{\boldsymbol{k}} \exp\{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}\} q_{\boldsymbol{k}} \tag{1.2}$$

where Ω is the volume of the system. Subsequently, a Heisenberg equation of motion can be calculated for it through a commutation relation with H_{eff}. Following standard techniques of quantum field theory (Jackiw 1977, Rajaraman 1987), the quantum field Ψ is decomposed into its classical component Φ (the so-called field translation) and a quantum correction Λ:

$$\Psi = \Phi + \Lambda. \tag{1.3}$$

Then, it can be shown that Φ satisfies a classical (nonlinear) equation of motion while Λ satisfies a linear Schrödinger equation in which Φ provides an effective potential.

The meaning of the classical component Φ is a field-theoretic analogue of the order parameter (Landau and Lifshitz 1959) and thus it plays a crucial role in describing the onset of criticality in the system.

It has been shown (Dixon and Tuszyński 1989) that in the regime of non-interacting particles $\Phi(\boldsymbol{x}, t)$ satisfies the linear equation

$$\mathrm{i}\hbar\partial_t\Phi = \nu_0\Phi + \mathrm{i}\boldsymbol{\nu}_1 \cdot (\nabla_\epsilon\Phi) - \tfrac{1}{2}\nabla_\epsilon^2\Phi \tag{1.4}$$

where ϵ represents the signatures of the space of independent variables. However, in the first non-trivial approximation (the so-called zeroth order) where the interaction parameter Δ is assumed to be a constant, the dynamics of Φ is described by the cubic nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\hbar\partial_t\Phi = \nu_0\Phi - \tfrac{1}{2}\nabla_\epsilon^2\Phi + \Omega f(\boldsymbol{\eta}_0, \boldsymbol{k}_0, \boldsymbol{m}_0)|\Phi|^2\Phi \tag{1.5}$$

where

$$f(\boldsymbol{\eta}, \boldsymbol{k}, \boldsymbol{m}) = 2\Delta_{\boldsymbol{\eta}+\boldsymbol{m}-\boldsymbol{k},\boldsymbol{k},\boldsymbol{m}}.$$

It should be pointed out that equation (1.4) as a linear equation is easy to solve (for example using integral transform methods), while in $1+1$ dimesions, i.e. one temporal and one spatial dimensions (so $\nabla_\epsilon^2 = \partial^2/\partial x^2$), equation (1.5) is the nonlinear Schrödinger equation (NLS) which is a well known example of a completely integrable soliton equation solvable by inverse scattering (Zakharov and Shabat 1972). In addition, in $3+1$ dimesions, equation (1.5) has been recently analysed in detail by Gagnon and Winternitz (1988, 1989a,b,c) and Gagnon *et al* (1989) using the method of symmetry reduction (cf Bluman and Cole 1974, Olver 1986, Bluman and Kumei 1989). These studies provided a large number of exact solutions with very interesting geometrical forms in three-dimensional space. These include axial, cylindrical, spherical, conical and even spiral types. Their functional dependence on the spatial and temporal variables, $\boldsymbol{x} = (x_1, x_2, x_3)$ and t respectively, is usually represented by Jacobi elliptic functions, often with a damping prefactor. In several cases these solutions can be regarded as localised when the limit of the Jacobi modulus k is taken with $k \to 1$. Physically, this behaviour may be identified with the phenomenon of coherence in the system (Klauder and Skagerstam 1985).

Going one step further one has to consider the interaction coefficients ω and Δ to be momentum-dependent which physically means that the strength of the interactions depends on the distance between the particles. As a result, the equation of motion for the order parameter field $\Phi(\boldsymbol{x}, t)$, ignoring quantum effects, becomes

$$\mathrm{i}\hbar\partial_t\Phi = \nu_0\Phi + \mathrm{i}\boldsymbol{\nu}_1 \cdot (\nabla_\epsilon\Phi) - \tfrac{1}{2}\nabla_\epsilon^2\Phi + \nu_2(\Phi^*\Phi)\Phi + 2\mathrm{i}\boldsymbol{\nu}_3 \cdot [\Phi^*\Phi(\nabla_\epsilon\Phi)]. \tag{1.6}$$

In the earlier work (Tuszyński and Dixon 1989) solutions to equation (1.6) have only been considered when $\Phi(\boldsymbol{x}, t)$ is represented as

$$\Phi(\boldsymbol{x}, t) = \eta(\boldsymbol{x}, t)\exp[\mathrm{i}\chi(\boldsymbol{x}, t)] \tag{1.7}$$

and when both the envelope $\eta(\boldsymbol{x}, t)$ and the carrier wave $\chi(\boldsymbol{x}, t)$ propagate in orthogonal directions to that

$$\nabla_\epsilon\eta \cdot \nabla_\epsilon\chi = 0. \tag{1.8}$$

This condition (and some additional constraints) allow one to cast the equation for the envelope η in the form of a nonlinear Klein–Gordon (NLKG) equation. As a result, carrier waves are found in the form of plane waves and the corresponding envelopes may either be elliptic plane or cylindrical waves propagating in the plane normal to the propagation direction of the carrier (Dixon and Tuszyński 1989). Again, the limit of $k \to 1$ leads to the onset of localisation manifested by the creation of solitary waves.

In the present paper we consider a more general version of equation (1.6). In the earlier work of Tuszyński and Dixon (1989) the full quantum equation in first order was:

$$\begin{aligned} \mathrm{i}\hbar\partial_t\Psi = {} & \nu_0\Psi + \mathrm{i}\boldsymbol{\nu}_1\cdot(\nabla_\epsilon\Psi) - \tfrac{1}{2}\nabla_\epsilon^2\Psi + \nu_2\Psi^\dagger\Psi\Psi + \mathrm{i}\boldsymbol{\nu}_3\cdot\Psi^\dagger\Psi(\nabla\Psi) \\ & + \mathrm{i}\boldsymbol{\nu}_4\cdot\Psi^\dagger(\nabla\Psi)\Psi + \boldsymbol{\nu}_5\cdot(\nabla\Psi^\dagger)\Psi\Psi. \end{aligned} \tag{1.9}$$

It was subsequently assumed that all two-particle collisions are perfectly elastic which meant the elimination of the last term (i.e. that $\boldsymbol{\nu}_5 = 0$). However, we need not restrict ourselves to this case since the mathematical techniques available allow us to obtain exact analytical solutions in a more general case and moreover, the solutions we seek are not be constrained by condition (1.8). Thus, in classical approximation equation (1.9) becomes

$$\mathrm{i}\hbar\partial_t\Phi = \nu_0\Phi+\mathrm{i}\boldsymbol{\nu}_1\cdot(\nabla_\epsilon\Phi)-\tfrac{1}{2}\nabla_\epsilon^2\Phi+\nu_2(\Phi^*\Phi)\Phi+\mathrm{i}(\boldsymbol{\nu}_3+\boldsymbol{\nu}_4)\cdot(\Phi^*\Phi)\nabla\Phi+\mathrm{i}\boldsymbol{\nu}_5\cdot(\nabla\Phi^*)\Phi^2 \tag{1.10}$$

and our interest is in obtaining exact analytical solutions of this equation. We do this by considering similarity reductions of equation (1.10) which reduce it an ordinary differential equation. Painlevé analysis (see below) is then frequently used to determine whether the resulting ordinary differential equation is of Painlevé type (i.e. its solutions have no movable singularities other than poles). It appears to be the case that whenever an ordinary differential equation arising from a similarity reduction of a given partial differential equation is of Painlevé type, then one can explicitly solve the ordinary differential equation and hence obtain exact solutions to the original partial differential equation; whilst if the ordinary differential equation is not of Painlevé type, then usually one cannot solve it explicitly (cf Gagnon *et al* 1989, Gagnon and Winternitz 1988, 1989a,b,c, Grundland *et al* 1987, Skierski *et al* 1988, 1989, Winternitz *et al* 1987, 1988 for examples).

The *Painlevé conjecture* (or *Painlevé* ODE *test*) as formulated by Ablowitz *et al* (1978, 1980) and Hastings and McLeod (1980) asserts that every ordinary differential equation which arises as a similarity reduction of a nonlinear partial differential equation solvable by inverse scattering is of *Painlevé type*, though perhaps only after a transformation of variables. Ablowitz *et al* (1980) and McLeod and Olver (1983) have given proofs of the Painlevé ODE test under certain restrictions. Subsequently, Weiss *et al* (1983) proposed the *Painlevé* PDE *test* as a method of applying the Painlevé ODE test directly to a given partial differential equation without having to consider similarity reductions (which might not exist anyway). As for the Painlevé ODE test, at present there is no rigorous proof of the Painlevé PDE test; though a partial proof can be inferred from the partial proof of the Painlevé ODE test due to McLeod and Olver (1983). Despite being by no means foolproof, the Painlevé tests appear to provide useful criteria for the identification of completely integrable partial differential equations.

Furthermore, in addition to providing a valuable first test for whether a given partial differential equation is completely integrable, other important information can be obtained by use of Painlevé analysis such as Bäcklund transformations, Lax pairs, Hirota's bilinear representation, special and rational solutions for completely integrable equations and special and rational solutions for non-integrable equations (Cariello and Tabor 1989, Chudnovsky *et al* 1983, Conte 1988, Conte and Musette 1989, Fournier *et al* 1988, Fournier and Spiegel 1987, Gibbon *et al* 1985, Gibbon *et al* 1988, Levine and Tabor 1988, Newell *et al* 1987, Nozaki 1987, Weiss 1983, 1984a,b, 1985a,b, 1986a,b, 1987).

2. Mathematical background

In $1+1$ dimensions, equation (1.10) reduces to

$$\mathrm{i}\hbar\Phi_t = \nu_0\Phi + \mathrm{i}\nu_1\Phi_x - \tfrac{1}{2}\Phi_{xx} + \nu_2\Phi^*\Phi^2 + \mathrm{i}(\nu_3+\nu_4)\Phi^*\Phi\Phi_x + \mathrm{i}\nu_5\Phi^2\Phi_x^* \tag{2.1}$$

where ν_0, ν_1, ν_2, ν_3, ν_4 and ν_5 are arbitrary constants, with either $\nu_3+\nu_4 \neq 0$ or $\nu_5 \neq 0$. If in (2.1) we make the transformation

$$\Phi(x,t) = u(\xi,\tau)\exp[-\mathrm{i}\nu_0 t/\hbar] \tag{2.2a}$$

with

$$\xi = x + \nu_1 t/\hbar \qquad \tau = -t/(2\hbar) \tag{2.2b}$$

then $u(\xi,\tau)$ satisfies

$$\mathrm{i}u_\tau = u_{\xi\xi} - 2\mathrm{i}(\nu_3+\nu_4)u^*uu_\xi - 2\mathrm{i}\nu_5 u^2u_\xi^* - 2\nu_2u^*u^2. \tag{2.3}$$

This equation is a special case of the generalised mixed nonlinear Schrödinger equation (GMNLS)

$$\mathrm{i}u_\tau = u_{\xi\xi} + \mathrm{i}au^*uu_\xi + \mathrm{i}bu^2u_\xi^* + cu^{*2}u^3 + du^*u^2 \tag{2.4}$$

with a, b, c and d real constants, which was discussed by Clarkson and Cosgrove (1987) and arises in the modulation of Stokes waves of uniform depth near the marginal state (see Johnson 1977, Kakutani and Michihiro 1983, Parkes 1987). The GMNLS additionally has as special cases both the generalised derivative nonlinear Schrödinger equation (GDNLS)

$$\mathrm{i}u_\tau = u_{\xi\xi} + \mathrm{i}au^*uu_\xi + \mathrm{i}bu^2u_\xi^* + cu^{*2}u^3. \tag{2.5}$$

which was also discussed by Clarkson and Cosgrove (1987), and the mixed nonlinear Schrödinger equation (MNLS) due to Wadati *et al* (1979)

$$\mathrm{i}u_\tau = u_{\xi\xi} + \mathrm{i}b(u^*u^2)_\xi + du^*u^2. \tag{2.6}$$

The MNLS (2.6) itself is a combination of two completely integrable soliton equations which are solvable by inverse scattering, namely the nonlinear Schrödinger equation

$$\mathrm{i}u_\tau = u_{\xi\xi} + du^*u^2 \tag{2.7}$$

(Zakharov and Shabat 1972), and the derivative nonlinear Schrödinger equation (DNLSI — see below for second and third cases)

$$\mathrm{i}u_\tau = u_{\xi\xi} + \mathrm{i}b(u^*u^2)_\xi \tag{2.8}$$

(Kaup and Newell 1978), though as shown below, MNLS is equivalent to DNLSI under a point transformation (see also Kundu 1984).

Clarkson and Cosgrove (1987) determined under what conditions the GMNLS (2.4) and GDNLS (2.5) might be completely integrable (i.e. solvable by inverse scattering). They showed that the Painlevé PDE test due to Weiss *et al* (1983) suggests that a necessary condition for both equations (2.4) and (2.5) to be completely integrable is that

$$c = \tfrac{1}{4}b(2b-a). \tag{2.9}$$

Under this condition, equations (2.4) and (2.5) respectively become

$$\mathrm{i}u_\tau = u_{\xi\xi} + \mathrm{i}au^*uu_\xi + \mathrm{i}bu^2u_\xi^* + \tfrac{1}{4}b(2b-a)u^{*2}u^3 + du^*u^2 \tag{2.10}$$

and

$$\mathrm{i}u_\tau = u_{\xi\xi} + \mathrm{i}au^*uu_\xi + \mathrm{i}bu^2u_\xi^* + \tfrac{1}{4}b(2b-a)u^{*2}u^3 \tag{2.11}$$

which Kundu (1984) calls the 'higher-order nonlinear Schrödinger equation'. Clearly, equation (2.11) contains DNLSI ($a = 2b$), a second derivative nonlinear Schrödinger equation (DNLSII) due to Chen *et al* (1979)

$$\mathrm{i}u_\tau = u_{\xi\xi} + \mathrm{i}au^*uu_\xi \tag{2.12}$$

and a third derivative nonlinear Schrödinger equation (DNLSIII) due to Gerdjikov and Ivanov (1982)

$$\mathrm{i}u_\tau = u_{\xi\xi} + \mathrm{i}bu^2u_\xi^* + \tfrac{1}{2}b^2u^{*2}u^3 \tag{2.13}$$

as special cases. However, GDNLS is not a non-trivial generalisation since if in (2.11) we make the U(1) gauge transformation due to Kundu (1984)

$$\hat{u}(\xi,\tau) = u(\xi,\tau)\exp(\mathrm{i}k\psi) \tag{2.14}$$

where k is a real parameter and the potential $\psi(\xi,\tau)$ is defined by

$$\psi_\xi = u^*u \tag{2.15a}$$

$$\psi_\tau = \mathrm{i}(uu_\xi^* - u^*u_\xi) + \tfrac{1}{2}(a+b)u^{*2}u^2 \tag{2.15b}$$

then $\hat{u}(\xi,\tau)$ satisfies

$$\mathrm{i}\hat{u}_\tau = \hat{u}_{\xi\xi} + \mathrm{i}\hat{a}\hat{u}^*\hat{u}\hat{u}_\xi + \mathrm{i}\hat{b}\hat{u}^2\hat{u}_\xi^* + \tfrac{1}{4}\hat{b}(2\hat{b}-\hat{a})\hat{u}^{*2}\hat{u}^3 \tag{2.16a}$$

where

$$\hat{a} = a - 2k \qquad \hat{b} = b - 2k. \tag{2.16b}$$

Here, choosing $k = \frac{1}{2}(2b - a)$, $k = \frac{1}{2}b$ and $k = \frac{1}{2}a$ gives DNLSI, DNLSII and DNLSIII, respectively. This shows that (2.11), DNLSI, DNLSII and DNLSIII are all equivalent. The equivalence of DNLSI and DNLSII apparently was first noticed by Wadati and Sogo (1983), although the transformation is implied in the work of Kaup and Newell (1978) — see their equations (11) and (12). Whilst DNLSII does not appear explicitly in Kaup and Newell (1978), their variables Q and R solve (2.12) with $Q = u^*$, $R = u$, $a = 1$. Wadati and Sogo (1983) also found the Miura-type Bäcklund transformation relating solutions of DNLSI, DNLSII and NLS (see Clarkson and Cosgrove (1987) for details in the present notation).

Kaup and Newell (1978) solved the initial value problems for DNLSI (2.8) with the boundary condition

$$u(\xi, \tau) \to 0 \qquad \text{as } |\xi| \to \infty \tag{2.17}$$

using an inverse scattering formalism of Zakharov–Shabat type. This has subsequently been extended to the boundary conditions

$$u(\xi, \tau) \to \text{constant} \qquad \text{as } |\xi| \to \infty \tag{2.18}$$

(Kawata and Inoue 1978) and

$$u(\xi, \tau) \to (\text{constant}) \exp[\mathrm{i}(k\xi - \omega\tau)] \qquad \text{as } |\xi| \to \infty \tag{2.19}$$

(Kawata *et al* 1980). Chen *et al* (1979) and Dodd and Fordy (1984) write down associated linear problems for DNLSII (2.12) but do not solve the equation. Hence Clarkson and Cosgrove (1987) concluded that condition (2.9) is at least sufficient for complete integrability and conjectured that GDNLS (2.5) is nonintegrable otherwise due to the nature of the singularities of $u(\xi, \tau)$ which arise in Painlevé analysis when (2.9) does not hold.

If in GMNLS (2.4) we make the Lie-point transformation

$$\tilde{u}(\tilde{\xi}, \tilde{\tau}) = u(\xi, \tau) \exp[-\mathrm{i}(\alpha\xi + \alpha^2\tau)] \tag{2.20a}$$

with

$$\tilde{\xi} = \xi + 2\alpha\tau \qquad \tilde{\tau} = \tau \tag{2.20b}$$

then $\tilde{u}(\tilde{\xi}, \tilde{\tau})$ satisfies an equation of the same form with

$$\tilde{a} = a \qquad \tilde{b} = b \qquad \tilde{c} = c \qquad \tilde{d} = d - \alpha(a - b). \tag{2.21}$$

This shows that by choosing $\alpha = d/(a-b)$, provided that $a \neq b$, the point transformation (2.20) maps the GMNLS (2.4) into the GDNLS (2.5). (In particular, as mentioned previously, the MNLS (2.6) is thus equivalent to DNLSI.) The transformation (2.20) was used by Kawata *et al* (1980) to solve the initial value problem for DNLSI by inverse scattering with the boundary condition (2.19), which was more general then the earlier treatments by Kaup and Newell (1978) and Kawata and Inoue (1978) with the boundary conditions (2.17) and (2.18), respectively. Additionally, the GMNLS (2.4) also admits the gauge transformation (2.14, 15), irrespective of the value of d, the new parameters being

$$\tilde{a} = a - 2k \qquad \tilde{b} = b - 2k \qquad \tilde{c} = c + \tfrac{1}{2}k(a - 3b) + k^2 \qquad \tilde{d} = d. \tag{2.22}$$

Hence if $a = b$ and $c = \frac{1}{4}b^2$ then GMNLS is equivalent to NLS, and so is integrable; whilst if $a = b$ and $c \neq \frac{1}{4}b^2$, then GMNLS does not pass the Painlevé PDE test, and so is probably not integrable. Consequently, Clarkson and Cosgrove (1987) concluded that (2.9) is also a sufficient condition for the GMNLS (2.4) to be completely integrable. Furthermore, they write down the associated linear problem for GMNLS under the restriction (2.9) (i.e. equation (2.10)), together with the effects of the gauge and point transformations, so that one can readily adapt the inverse scattering formalisms of Kaup and Newell (1978), Kawata and Inoue (1978) and Kawata *et al* (1980) to any of the integrable members in the GMNLS or GDNLS families.

These results show that equation (2.3) (and hence also (2.1)) is completely integrable only for two special cases namely $\nu_3 + \nu_4 = 2\nu_5$ (in which case (2.3) is the MNLS and so is equivalent to DNLSI) and $\nu_5 = 0$ (when (2.3) is equivalent to DNLSII).

We shall now discuss the integrability of equation (1.10) in 3 + 1 dimesions, with either $\boldsymbol{\nu}_3 + \boldsymbol{\nu}_4 \neq \mathbf{0}$ or $\boldsymbol{\nu}_5 \neq \mathbf{0}$. First in (1.10) we make the transformation

$$\Phi(\boldsymbol{x}, t) = u(\boldsymbol{\xi}, \tau) \exp[-\mathrm{i}\nu_0 t/\hbar] \tag{2.23a}$$

with

$$\boldsymbol{\xi} = \boldsymbol{x} - (t/\hbar)\boldsymbol{\nu}_1 \qquad \tau = t/(2\hbar) \tag{2.23b}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$. Then, $u(\boldsymbol{\xi}, \tau)$ satisfies

$$\mathrm{i}u_t = -\nabla^2 u + 2\mathrm{i}(\boldsymbol{\nu}_3 + \boldsymbol{\nu}_4) \cdot (u^* u \nabla u) + 2\mathrm{i}\boldsymbol{\nu}_5 \cdot (u^2 \nabla u^*) + 2\nu_2 u^* u^2 \tag{2.24}$$

where $\boldsymbol{\nu}_k = (\nu_{k_1}, \nu_{k_2}, \nu_{k_3})$ for $k = 3, 4, 5$ (this shows that if both $\boldsymbol{\nu}_3 + \boldsymbol{\nu}_4 = \mathbf{0}$ and $\boldsymbol{\nu}_5 = \mathbf{0}$ then (1.10) is equivalent to the three-dimensional nonlinear Schrödinger equation). Now suppose that

$$u(\boldsymbol{\xi}, \tau) = U(\eta, \tau) \qquad \eta = \boldsymbol{\lambda} \cdot \boldsymbol{\xi} \equiv \lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3$$

with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ an arbitrary constant vector, then $U(\eta, \tau)$ satisfies

$$\mathrm{i}U_\tau = -U_{\eta\eta} + 2\mathrm{i}(\gamma_3 + \gamma_4)U^* U U_\eta + 2\mathrm{i}\gamma_5 U^2 U^*_\eta + 2\nu_2 U^* U^2 \tag{2.25}$$

which is the same as equation (2.3). It follows from the results for equation (2.3) that a necessary condition for (2.25) to be completely integrable is that

$$(\gamma_3 + \gamma_4 - 2\gamma_5)\gamma_5 = 0 \qquad j = 1, 2, 3. \tag{2.26}$$

with $\gamma_k = \boldsymbol{\lambda} \cdot \boldsymbol{\nu}_k$, for $k = 3, 4, 5$. If otherwise then equation (2.25) is reducible to an ordinary differential equation which is not of Painlevé type (since it possesses solutions which have movable logarithmic branch points — see Clarkson and Cosgrove 1987). Therefore, the Painlevé ODE test due to Ablowitz *et al* (1978, 1980) asserts that a necessary condition for equation (2.24) to be completely integrable is that either

$$\boldsymbol{\nu}_3 + \boldsymbol{\nu}_4 - 2\boldsymbol{\nu}_5 = \mathbf{0} \tag{2.27}$$

or

$$\boldsymbol{\nu}_5 = \mathbf{0} \tag{2.28}$$

since if neither (2.27) nor (2.28) holds, then one can choose $\boldsymbol{\lambda}$ such that (2.26) also does not hold (e.g., if $(\boldsymbol{\nu}_3+\boldsymbol{\nu}_4-2\boldsymbol{\nu}_5)\cdot\boldsymbol{\nu}_5=0$, take $\boldsymbol{\lambda}=\boldsymbol{\nu}_3+\boldsymbol{\nu}_4-\boldsymbol{\nu}_5$, otherwise, take $\boldsymbol{\lambda}=\boldsymbol{\nu}_5$). If either (2.27) or (2.28) holds, then in (2.24) we can assume, without loss of generality, that $\nu_2=0$ since, if otherwise, we make the point transformation

$$u(\boldsymbol{\xi},\tau)=\tilde{u}(\tilde{\boldsymbol{\xi}},\tilde{\tau})\exp\left[\mathrm{i}\left(\frac{\nu_2[(\boldsymbol{\nu}_3+\boldsymbol{\nu}_4-\boldsymbol{\nu}_5)\cdot\boldsymbol{\xi}+t]}{|\boldsymbol{\nu}_3+\boldsymbol{\nu}_4-\boldsymbol{\nu}_5|^2}\right)\right] \tag{2.29a}$$

with

$$\tilde{\boldsymbol{\xi}}=\boldsymbol{\xi}-\frac{\nu_2 t(\boldsymbol{\nu}_3+\boldsymbol{\nu}_4-\boldsymbol{\nu}_5)}{\hbar|\boldsymbol{\nu}_3+\boldsymbol{\nu}_4-\boldsymbol{\nu}_5|^2}\qquad \tilde{\tau}=\tau. \tag{2.29b}$$

(More generally, this transformation maps equation (2.24) into an equation of the same form with $\nu_2=0$, provided that $\boldsymbol{\nu}_3+\boldsymbol{\nu}_4\neq\boldsymbol{\nu}_5$.) Hence we have only two possible candidates for integrability, namely

$$\mathrm{i}u_\tau=-\nabla^2u+2\mathrm{i}(\boldsymbol{\nu}_3+\boldsymbol{\nu}_4)\cdot\nabla(u^*u^2) \tag{2.30}$$

$$\mathrm{i}u_\tau=-\nabla^2u+2\mathrm{i}(\boldsymbol{\nu}_3+\boldsymbol{\nu}_4)\cdot(u^*u\nabla u) \tag{2.31}$$

which are (3 + 1)-dimensional generalisations of DNLSI and DNLSII, respectively.

To discuss the integrability of equations (2.30, 31) consider the equation

$$\mathrm{i}u_\tau=-\nabla^2u+\mathrm{i}\boldsymbol{a}\cdot(u^*u\nabla u)+\mathrm{i}\boldsymbol{b}\cdot(u^2\nabla u^*). \tag{2.32}$$

Setting $u(\boldsymbol{\xi},\tau)=R(\boldsymbol{\xi})\exp[\mathrm{i}(\theta(\boldsymbol{\xi})-\mu\tau)]$ yields the coupled system

$$2\nabla R\cdot\nabla\theta+R\nabla^2\theta-(\boldsymbol{a}+\boldsymbol{b})\cdot(R^2\nabla R)=0 \tag{2.33a}$$

$$\nabla^2R=R(\nabla\theta)^2-\mu R+(\boldsymbol{b}-\boldsymbol{a})\cdot(R^3\nabla\theta). \tag{2.33b}$$

From (2.33a) we have

$$\nabla\theta=\tfrac{1}{4}R^2(\boldsymbol{a}+\boldsymbol{b})+(\nabla\phi)/R^2 \tag{2.34}$$

where ϕ is any solution of Laplace's equation $\nabla^2\phi=0$ and then (2.33b) becomes

$$\nabla^2R=\tfrac{1}{16}(5\boldsymbol{b}^2+2\boldsymbol{a}\cdot\boldsymbol{b}-3\boldsymbol{a}^2)R^5+[\mu+\tfrac{1}{2}(3\boldsymbol{b}-\boldsymbol{a})\cdot\nabla\phi]R+(\nabla\phi)^2/R^3. \tag{2.35}$$

In the case of spherical symmetry, $R=R(r)$ with $r=\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}$, this reduces to

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r}=\frac{1}{16}(5\boldsymbol{b}^2+2\boldsymbol{a}\cdot\boldsymbol{b}-3\boldsymbol{a}^2)R^5+\left[\mu+\frac{1}{2}(3\boldsymbol{b}-\boldsymbol{a})\cdot\nabla\phi\right]R+\frac{(\nabla\phi)^2}{R^3}. \tag{2.36}$$

It is easily shown that the equation

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r}=\alpha R^5+\beta R+\frac{\gamma}{R^3} \tag{2.37}$$

with α, β and γ constants such that either $\alpha\neq0$ or $\gamma\neq0$, is not of Painlevé type and so, in general (2.36) is not of Painlevé type either. Hence the Painlevé ODE test predicts that neither of equations (2.30) and (2.31) can be completely integrable. This together with the earlier results show that the Painlevé tests suggest that, in 3 + 1 dimesions, equation (1.10) is not completely integrable. However, the equation may be regarded as being 'partially integrable' since some special reductions are integrable. In the following section we present some such reductions and their explicit solutions.

3. Reductions resulting in explicit solutions

3.1. Propagating wave solutions

The starting point in search for new reductions is equation (1.10). We first look for solutions in the form of plane waves, both for the carrier and the envelope, i.e. we assume that:

$$\Phi(\boldsymbol{x},t) = R(\boldsymbol{x},t)\exp[\mathrm{i}\Theta(\boldsymbol{x},t)] \tag{3.1}$$

where

$$R(\boldsymbol{x},t) = R(\xi) \qquad \Theta(\boldsymbol{x},t) = \theta(\xi) - \lambda t \tag{3.2}$$

with

$$\xi = \boldsymbol{n}\cdot\boldsymbol{x} - \omega t \equiv n_1x_1 + n_2x_2 + n_3x_3 - \omega t \tag{3.3}$$

and $\boldsymbol{n} = (n_1, n_2, n_3)$ a unit vector describing the direction of propagation. The first step is to substitute equation (3.1) into equation (1.10) and decouple the real and imaginary parts to get

$$\hbar R_t = \boldsymbol{\nu}_1\cdot\nabla R - \tfrac{1}{2}(2\nabla R\cdot\nabla\Theta + R\nabla^2\Theta) + (\boldsymbol{\nu}_3 + \boldsymbol{\nu}_4 + \boldsymbol{\nu}_5)\cdot(R^2\nabla R) \tag{3.4}$$

for the imaginary part, and

$$-\hbar R\Theta_t = \nu_0 R - \boldsymbol{\nu}_1\cdot(R\nabla\Theta) - \tfrac{1}{2}\nabla^2 R + \tfrac{1}{2}R(\nabla\Theta)^2 + \nu_2R^3 - (\boldsymbol{\nu}_3 + \boldsymbol{\nu}_4 - \boldsymbol{\nu}_5)\cdot(R^3\nabla\Theta) \tag{3.5}$$

for the real part. Then, making use of equations (3.2) and (3.3) for the present reduction yields two coupled ordinary differential equations

$$-\omega\hbar R' = \mu_1R' - (R'\theta' + \tfrac{1}{2}R\theta'') + (\mu_3 + \mu_4 + \mu_5)R^2R' \tag{3.6}$$

$$-\hbar(\omega\theta' + \lambda)R = \nu_0R - \mu_1R\theta' - \tfrac{1}{2}R'' + \tfrac{1}{2}R(\theta')^2 + \nu_2R^3 + (\mu_5 - \mu_3 - \mu_4)R^3\theta' \tag{3.7}$$

with $' := \mathrm{d}/\mathrm{d}\xi$ and $\mu_j = \boldsymbol{n}\cdot\boldsymbol{\nu}_j$ for $j = 1, 3, 4, 5$. Equation (3.6) can be rearranged and integrated once to give

$$\theta'(\xi) = \tfrac{1}{2}(\mu_3 + \mu_4 + \mu_5)R^2 + (\mu_1 + \omega\hbar) + C/R^2 \tag{3.8}$$

with C an integration constant. Upon substituting equation (3.8) into equation (3.7) we obtain a second-order ordinary differential equation in terms of R only, i.e.

$$\begin{aligned}\tfrac{1}{2}R''(\xi) = {}& \tfrac{1}{8}(\mu_3 + \mu_4 + \mu_5)(5\mu_5 - 3\mu_3 - 3\mu_4)R^5 + [\nu_2 + (\mu_5 - \mu_3 - \mu_4)(\mu_1 + \omega\hbar)]R^3 \\ & + [\tfrac{1}{2}C(3\mu_5 - \mu_3 - \mu_4) + (\nu_0 - \hbar\lambda) - \tfrac{1}{2}(\mu_1 + \hbar\omega)^2]R + \tfrac{1}{2}C^2/R^3.\end{aligned} \tag{3.9}$$

This equation can be readily integrated once to yield

$$\begin{aligned}\tfrac{1}{4}(R')^2 = {}& \tfrac{1}{48}(\mu_3 + \mu_4 + \mu_5)(5\mu_5 - 3\mu_3 - 3\mu_4)R^6 + \tfrac{1}{4}[\nu_2 + (\mu_5 - \mu_3 - \mu_4)(\mu_1 + \omega\hbar)]R^4 \\ & + \tfrac{1}{2}[\tfrac{1}{2}C(3\mu_5 - \mu_3 - \mu_4) + (\nu_0 - \hbar\lambda) - \tfrac{1}{2}(\mu_1 + \hbar\omega)^2]R^2 + D - \tfrac{1}{4}C^2/R^2\end{aligned} \tag{3.10}$$

Table 1. Solutions of the equation: $(\mathrm{d}W/\mathrm{d}\xi)^2 = \epsilon(W-a)(W-b)(W-c)(W-d)$ where $\epsilon = \pm 1$ and a, b, c, d may be real or complex.

	Solution and corresponding figure		Condition
$\epsilon = -1$	$W = \dfrac{d(a-c) + a(c-d)\,\mathrm{sn}^2(\xi/g;k)}{(a-c)+(c-d)\,\mathrm{sn}^2(\xi/g;k)}$	Figure 1(a)	$d < W \le c < b < a$ $g = 2/[(a-c)(b-d)]^{1/2}$ $k^2 = \dfrac{(a-b)(c-d)}{(a-c)(b-d)}$
	$W = \dfrac{c(b-d) - b(c-d)\,\mathrm{sn}^2(\xi/g;k)}{(b-d)-(c-d)\,\mathrm{sn}^2(\xi/g;k)}$	Figure 1(b)	$d \le W < c < b < a$ $g = 2/[(a-c)(b-d)]^{1/2}$ $k^2 = \dfrac{(a-b)(c-d)}{(a-c)(b-d)}$
	$W = \dfrac{b(a-c) - c(a-b)\,\mathrm{sn}^2(\xi/g;k)}{(a-c)-(a-b)\,\mathrm{sn}^2(\xi/g;k)}$	Figure 1(c)	$d < c < b < W \le a$ $g = 2/[(a-c)(b-d)]^{1/2}$ $k^2 = \dfrac{(a-b)(c-d)}{(a-c)(b-d)}$
	$W = \dfrac{a(b-d) + d(a-b)\,\mathrm{sn}^2(\xi/g;k)}{(b-d)+(a-b)\,\mathrm{sn}^2(\xi/g;k)}$	Figure 1(d)	$d < c < b \le W < a$ $g = 2/[(a-c)(b-d)]^{1/2}$ $k^2 = \dfrac{(a-b)(c-d)}{(a-c)(b-d)}$
	$W = \dfrac{aB + bA + (bA - aB)\,\mathrm{cn}(\xi/g;k)}{A+B+(A-B)\,\mathrm{cn}(\xi/g;k)}$	Figure 1(e)	$b < a \qquad c, d = c^*\epsilon$ $A^2 = (a - \mathrm{Re}\,c)^2 + (\mathrm{Im}\,c)^2$ $B^2 = (b - \mathrm{Re}\,c)^2 + (\mathrm{Im}\,c)^2$ $g = 1/(AB)^{1/2}$ $k^2 = \dfrac{(a-b)^2 - (A-B)^2}{4AB}$

with D an integration constant. To simplify the analysis of its solutions further, we make the substitution $W = R^2$ which results in a standard elliptic form

$$(W')^2 = \tfrac{1}{3}(\mu_3+\mu_4+\mu_5)(5\mu_5 - 3\mu_3 - 3\mu_4)W^4 + 4[\nu_2 + (\mu_5-\mu_3-\mu_4)(\mu_1+\hbar\omega)]W^3 + 4[C(3\mu_5-\mu_3-\mu_4) + 2(\nu_0 - \hbar\lambda) - (\mu_1+\hbar\omega)^2]W^2 + 16DW - 4C^2. \tag{3.11}$$

In table 1 we have listed all the solutions to the above equation type and illustrated them with graphical representations in figure 1. It should be noted, however, that strictly speaking $R \ge 0$ and consequently $W \ge 0$, so that one must take $|W|$ for solutions in table 1 which are obtained when W is of arbitrary sign (but real). Another comment concerning solutions in table 1 is that virtually all of them are expressed in terms of Jacobi elliptic functions (cf Whittaker and Watson 1927). Depending on the number and location of real roots of the quartic polynomial on the right-hand side of equation (3.11) the solutions may be singular or regular. If a particular solution corresponds to only one real root, then it is singular. If, on the other hand, it oscillates

Table 1. (Continued)

	Solution and corresponding figure		Condition
$\epsilon = +1$	$W = \dfrac{d(a-c) - c(a-d)\,\mathrm{sn}^2(\xi/g;k)}{(a-c)-(a-d)\,\mathrm{sn}^2(\xi/g;k)}$	Figure 1(f)	$W < d < c < b < a$ $g = 2/[(a-c)(b-d)]^{1/2}$ $k^2 = \dfrac{(a-d)(b-c)}{(a-c)(b-d)}$
	$W = \dfrac{c(b-d) - d(b-c)\,\mathrm{sn}^2(\xi/g;k)}{(b-d)-(b-c)\,\mathrm{sn}^2(\xi/g;k)}$	Figure 1(g)	$d < c < W \leq b < a$ $g = 2/[(a-c)(b-d)]^{1/2}$ $k^2 \dfrac{(a-d)(b-c)}{(a-c)(b-d)}$
	$W = \dfrac{b(a-c) - a(b-c)\,\mathrm{sn}^2(\xi/g;k)}{(a-c)-(b-c)\,\mathrm{sn}^2(\xi/g;k)}$	Figure 1(h)	$d < c \leq W < b < a$ $g = 2/[(a-c)(b-d)]^{1/2}$ $k^2 \dfrac{(a-d)(b-c)}{(a-c)(b-d)}$
	$W = \dfrac{a(b-d) - b(a-d)\,\mathrm{sn}^2(\xi/g;k)}{(b-d)-(a-d)\,\mathrm{sn}^2(\xi/g;k)}$	Figure 1(i)	$d < c < b < a < W$ $g = 2/[(a-c)(b-d)]^{1/2}$ $k^2 \dfrac{(a-d)(b-c)}{(a-c)(b-d)}$
	$W = \dfrac{(aB-bA) + (aB+bA)\,\mathrm{cn}(\xi/g;k)}{(B-A)+(A+B)\,\mathrm{cn}(\xi/g;k)}$	Figure 1(j)	$b < a < W$ $A^2 = (a - \mathrm{Re}\,c)^2 + (\mathrm{Im}\,c)^2$ $B^2 = (b - \mathrm{Re}\,c)^2 + (\mathrm{Im}\,c)^2$ $g = 1/(AB)^{1/2}$ $k^2 = \dfrac{(A+B)^2 - (a-b)^2}{4AB}$

between two real roots, it is regular. In the limit of the Jacobi modulus $k \to 1$, the real solutions tend to solitary waves since

$$\mathrm{sn}(x,1) = \tanh x \qquad \text{and} \qquad \mathrm{cn}(x,1) = \mathrm{sech}\, x.$$

Due to the requirement that $R \geq 0$, some of the kink solutions (when both turning points are positive) will remain of kink type. Those, however, which involve two turning points of opposite sign will become cusp-like.

3.2. Scaling reduction

In section 2 it has been demonstrated that, without loss of generality, the equation of motion, equation (1.10), can be transformed into a simpler-looking equation, i.e. equation (2.32), which will be our starting point in the search for scale-invariant reductions. (We shall assume that either $\boldsymbol{a} \neq \mathbf{0}$ or $\boldsymbol{b} \neq \mathbf{0}$ in equation (2.32) since, if otherwise, then it is just a linear equation.) Representing the dependent variable u in Euler form

$$u(\boldsymbol{\xi}, \tau) = R(\boldsymbol{\xi}, \tau)\exp[\mathrm{i}\Theta(\boldsymbol{\xi}, \tau)] \tag{3.12}$$

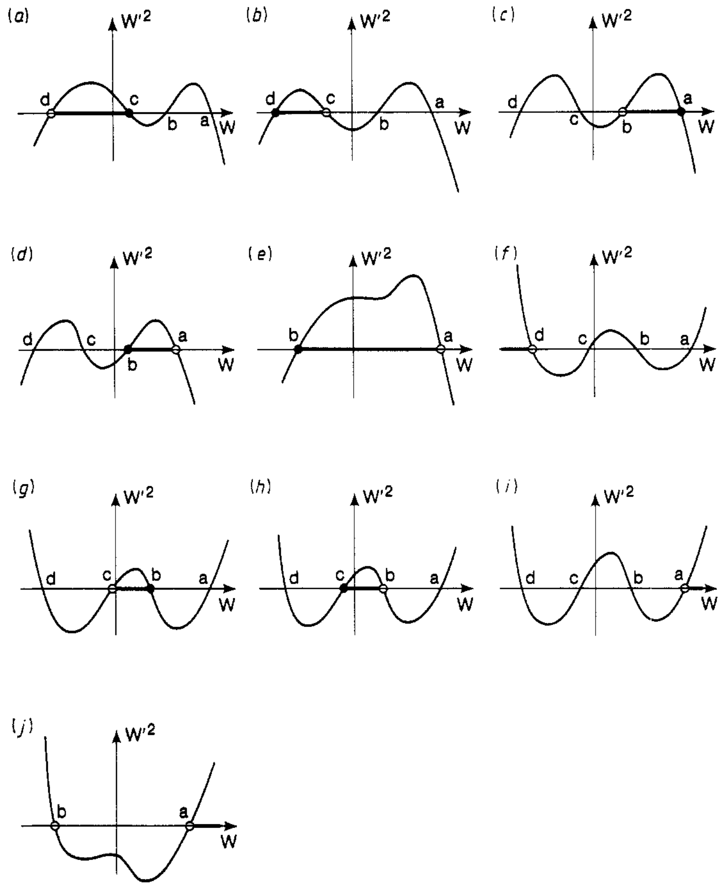

Figure 1. Plots of $(W')^2$ against W for solutions of the equation $(W')^2 = \epsilon(W - a)(W - b)(W - c)(W - d)$ corresponding to table 1.

and separating the real and imaginary parts gives the following two coupled partial differential equations

$$R_\tau = -2\nabla R \cdot \nabla\Theta - R\nabla^2\Theta + (\boldsymbol{a}+\boldsymbol{b})\cdot(R^2\nabla R) \tag{3.13}$$

$$-R\Theta_\tau = -\nabla^2 R + R(\nabla\Theta)^2 + (\boldsymbol{b}-\boldsymbol{a})\cdot(R^3\nabla\Theta). \tag{3.14}$$

We then look for $R(\boldsymbol{\xi},\tau)$ and $\Theta(\boldsymbol{\xi},\tau)$ in the form

$$R(\boldsymbol{\xi},\tau) = \tau^{-1/4}R(\eta) \tag{3.15}$$

$$\Theta(\boldsymbol{\xi},\tau) = \theta(\eta) + \gamma\ln\tau \tag{3.16}$$

with γ a constant and where the new independent variable η (the symmetry variable) is given by

$$\eta = (\boldsymbol{n}\cdot\boldsymbol{\xi})/\tau^{1/2} \equiv (n_1\xi_1 + n_2\xi_2 + n_3\xi_3)/\tau^{1/2} \tag{3.17}$$

with $\boldsymbol{n}^2 = 1$. Substituting (3.15)–(3.17) into (3.13) and (3.14) results in two coupled ordinary differential equations in terms of R and θ, i.e.

$$-\left(\tfrac{1}{4}R + \tfrac{1}{2}\eta R'\right) = -2R'\theta' - R\theta'' + (\boldsymbol{a}\cdot\boldsymbol{n} + \boldsymbol{b}\cdot\boldsymbol{n})R^2R' \tag{3.18}$$

$$\left(\tfrac{1}{2}\eta R\theta' - \gamma R\right) = -R'' + R(\theta')^2 + (\alpha - \beta)R^3\theta' \tag{3.19}$$

with $' := \mathrm{d}/\mathrm{d}\eta$ and $\alpha := \boldsymbol{a}\cdot\boldsymbol{n}$ and $\beta := \boldsymbol{b}\cdot\boldsymbol{n}$. Equation (3.18) can be recast as

$$(R^2\theta')' = \tfrac{1}{4}(\eta R^2)' + \tfrac{1}{4}(\beta + \alpha)(R^4)' \tag{3.20}$$

whence

$$\theta'(\eta) = \tfrac{1}{4}\eta + \tfrac{1}{4}(\beta + \alpha)R^2 + C/R^2 \tag{3.21}$$

where C is an arbitrary integration constant. This can then be substituted into equation (3.19) to produce a non-autonomous ordinary differential equation in terms of $R(\eta)$ only

$$R''(\eta) = \tfrac{1}{16}(\beta + \alpha)(5\alpha - 3\beta)R^5 + \tfrac{1}{4}(\alpha - \beta)\eta R^3 + [\gamma + \tfrac{1}{2}C(3\beta - \alpha) - \tfrac{1}{16}\eta^2]R + C^2/R^3. \tag{3.22}$$

There are two cases to consider.

Case 1. $\alpha \neq \beta$ (i.e. $\boldsymbol{a}\cdot\boldsymbol{n} \neq \boldsymbol{b}\cdot\boldsymbol{n}$). In this case the transformation

$$R^2(\eta) = \frac{1 + \mathrm{i}}{\sqrt{2}\,(\alpha - \beta)}\,W(Z) \qquad \eta = \sqrt{2}\,(1 + \mathrm{i})\,Z \tag{3.23}$$

maps equation (3.22) into

$$\frac{\mathrm{d}^2W}{\mathrm{d}Z^2} = \frac{1}{2W}\left(\frac{\mathrm{d}W}{\mathrm{d}Z}\right)^2 + \frac{(\alpha + \beta)(3\alpha - 5\beta)}{2(\alpha - \beta)^2}W^3 + 4ZW^2 + 2(Z^2 - A)W + \frac{B}{W} \tag{3.24a}$$

with

$$A = -\mathrm{i}[4\gamma + 2C(3\beta - \alpha)] \qquad \text{and} \qquad B = 4\sqrt{2}\,(1 + \mathrm{i})\,C^2(\alpha - \beta). \tag{3.24b}$$

It is easily shown that a necessary and sufficient condition for equation (3.24) to be of Painlevé type is $\beta(\alpha - 2\beta) = 0$, i.e. if and only if $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{n}$ satisfy the constraint

$$\boldsymbol{b}\cdot\boldsymbol{n}(\boldsymbol{a}\cdot\boldsymbol{n} - 2\boldsymbol{b}\cdot\boldsymbol{n}) = 0. \tag{3.25}$$

If this condition holds then (3.24) is the fourth Painlevé (PIV) equation (cf Ince 1956)

$$\frac{\mathrm{d}^2W}{\mathrm{d}Z^2} = \frac{1}{2W}\left(\frac{\mathrm{d}W}{\mathrm{d}Z}\right)^2 + \tfrac{3}{2}W^3 + 4ZW^2 + 2(Z^2 - A)W + \frac{B}{W} \tag{3.26}$$

with A, B arbitrary constants (since γ and C are arbitrary); otherwise solutions of (3.24) possess movable logarithmic branch points. Clearly, if either $\boldsymbol{b} = 0$ or $\boldsymbol{a} = 2\boldsymbol{b}$ then (3.25) holds for all $\boldsymbol{n}$. Otherwise, we require that either

$$\boldsymbol{n} = \frac{\boldsymbol{x}\wedge(\boldsymbol{a} - 2\boldsymbol{b})}{|\boldsymbol{x}\wedge(\boldsymbol{a} - 2\boldsymbol{b})|} \qquad \text{or} \qquad \boldsymbol{n} = \frac{\boldsymbol{x}\wedge\boldsymbol{b}}{|\boldsymbol{x}\wedge\boldsymbol{b}|} \tag{3.27}$$

where $\boldsymbol{x}$ is any vector such that $\boldsymbol{x} \wedge (\boldsymbol{a} - 2\boldsymbol{b}) \neq \mathbf{0}$ or $\boldsymbol{x} \wedge \boldsymbol{b} \neq \mathbf{0}$ respectively, for the constraint (3.25) to be satisfied (recall $\boldsymbol{n}$ is a unit vector).

To summarise, for all choices of $\boldsymbol{a}$ and $\boldsymbol{b}$, there exist choices of $\boldsymbol{n}$ given by (3.27) such that equation (3.22) may be transformed into PIV. Conversely, for all $\boldsymbol{a}$ and $\boldsymbol{b}$, unless either $\boldsymbol{b} = \mathbf{0}$ or $\boldsymbol{a} = 2\boldsymbol{b}$, there exist vectors $\hat{\boldsymbol{n}}$ such that (3.22) is not of Painlevé type. (Note that if either $\boldsymbol{b} = \mathbf{0}$ or $\boldsymbol{a} = 2\boldsymbol{b}$, then equation (3.22) may be transformed into PIV for all choices $\boldsymbol{n}$.) Since there exist rational solutions and one-parameter families of solutions of PIV (cf Airault 1979, Fokas and Ablowitz 1983, Gibbon *et al* 1988, Clarkson 1990), then we can obtain exact analytical solutions for equation (2.32) through this scaling similarity reduction.

Case 2. $\alpha = \beta$ (i.e. $\boldsymbol{a} \cdot \boldsymbol{n} = \boldsymbol{b} \cdot \boldsymbol{n}$). In this case the transformation

$$R^2(\eta) = \kappa W(Z) \qquad \eta = \sqrt{2}(1+\mathrm{i})Z \tag{3.28}$$

maps equation (3.22) into

$$\frac{\mathrm{d}^2 W}{\mathrm{d}Z^2} = \frac{1}{2W}\left(\frac{\mathrm{d}W}{\mathrm{d}Z}\right)^2 - 2\mathrm{i}\kappa^2\alpha^2 W^3 + 2(Z^2 - A)W + \frac{B}{W} \tag{3.29a}$$

with

$$A = -4\mathrm{i}[\gamma + C\alpha] \qquad B = 4\sqrt{2}(1+\mathrm{i})C^2/\kappa^2. \tag{3.29b}$$

It is easily shown that this equation is of Painlevé type if and only if $\alpha = 0$, i.e. if and only if

$$\boldsymbol{a} \cdot \boldsymbol{n} = 0 \tag{3.30}$$

whence equations (3.18) and(3.19) simplify to

$$\tfrac{1}{4}R + \tfrac{1}{2}\eta R' = 2R'\theta' + R\theta'' \tag{3.31}$$

$$\tfrac{1}{2}\eta R\theta' - \gamma R = -R'' + R(\theta')^2. \tag{3.32}$$

(We remark that these equations also arise from the linearised version of equations (2.32), i.e.

$$\mathrm{i}u_\tau = -\nabla^2 u \tag{3.33}$$

through the scaling reduction

$$u(\boldsymbol{\xi}, \tau) = \tau^{-1/4} R(\eta) \exp\{\mathrm{i}[\theta(\eta) + \gamma \ln \tau]\} \qquad \eta = (\boldsymbol{n} \cdot \boldsymbol{\xi})/\tau^{1/2} \tag{3.34}$$

with γ an arbitary constant and $\boldsymbol{n}$ an arbitrary unit vector.) Therefore for equation (3.29) to be of Painlevé-type, we require that if $\boldsymbol{a} \wedge \boldsymbol{b} \neq \mathbf{0}$ then $\boldsymbol{n}$ has the form

$$\boldsymbol{n} = \frac{\boldsymbol{a} \wedge \boldsymbol{b}}{|\boldsymbol{a} \wedge \boldsymbol{b}|}$$

or if $\boldsymbol{a} \wedge \boldsymbol{b} = \mathbf{0}$ then $\boldsymbol{n}$ has one of the forms

$$\boldsymbol{n} = \frac{\boldsymbol{x} \wedge \boldsymbol{a}}{|\boldsymbol{x} \wedge \boldsymbol{a}|} \qquad \text{if } \boldsymbol{a} \neq 0$$

$$\boldsymbol{n} = \frac{\boldsymbol{x} \wedge \boldsymbol{b}}{|\boldsymbol{x} \wedge \boldsymbol{b}|} \qquad \text{if } \boldsymbol{b} \neq 0$$

where $\boldsymbol{x}$ is any vector such that $\boldsymbol{x} \wedge \boldsymbol{a} \neq \mathbf{0}$ or $\boldsymbol{x} \wedge \boldsymbol{b} \neq \mathbf{0}$ respectively, in order that the constraint (3.30) to be satisfied.

4. Phenomenological equations with currents

Tuszyński and Dixon (1989) demonstrated that the zeroth-order approximation in the field theoretic treatment of the second-quantised Hamiltonian (1.1) corresponds to the Landau–Ginzburg–Wilson Hamiltonian density, i.e.

$$H = A_2|\Phi|^2 + A_4|\Phi|^4 + D|\nabla\Phi|^2. \tag{4.1}$$

In fact, in an earlier paper, Otwinowski *et al* (1986) considered a corresponding Lagrangian density in the presence of currents flowing through the system and classical kinetic energy so that

$$\mathcal{L} = \tfrac{1}{2}\mathrm{i}\kappa(\Phi_t\Phi^* - \Phi\Phi_t^*) + \tfrac{1}{2}\mathrm{i}\boldsymbol{\mu}\cdot[(\nabla\Phi)\Phi^* - \Phi(\nabla\Phi^*)] + K|\Phi_t|^2 - D|\nabla\Phi|^2 - A_2|\Phi|^2 - A_4|\Phi|^4 \tag{4.2}$$

where K is associated with the classical effective mass of the system, κ and μ are the external fields coupled to the two current densities. The latter quantities are defined through

$$j^0 = \mathrm{i}K(\Phi^*\Phi_t - \Phi\Phi_t^*) \tag{4.3}$$

$$j^1 = \mathrm{i}K(\Phi^*\nabla\Phi - \Phi\nabla\Phi^*). \tag{4.4}$$

The Hamiltonian density that is obtained directly from $\mathcal{L}$ is

$$H = \tfrac{1}{2}\mathrm{i}\boldsymbol{\mu}\cdot[\Phi(\nabla\Phi^*) - \Phi^*(\nabla\Phi)] + K|\Phi_t|^2 + D|\nabla\Phi|^2 + A_2|\Phi|^2 + A_4|\Phi|^4 \tag{4.5}$$

and it is easy to verify that the extra term originates from the second-quantised Hamiltonian through the off-diagonal interactions of the form

$$\omega_{k,l}a_k^\dagger a_l$$

where $k \neq l$. In their paper, Otwinowski *et al* (1986) only considered solitary wave solutions of the corresponding $(1+1)$-dimensional Euler–Lagrange equation of motion for Φ, i.e.

$$\mathrm{i}\kappa\Phi_t + \mathrm{i}\mu\Phi_x + D\Phi_{xx} - K\Phi_{tt} - A_2\Phi - A_4|\Phi|^2\Phi = 0 \tag{4.6}$$

which they called this equation the nonlinear Dirac–Klein–Gordon equation.

In this paper, we consider the $(3+1)$-dimensional analogue of equation (4.6), namely

$$\mathrm{i}\kappa\Phi_t + \mathrm{i}\boldsymbol{\mu}\cdot\nabla\Phi + D\nabla^2\Phi - K\Phi_{tt} - A_2\Phi - A_4|\Phi|^2\Phi = 0. \tag{4.7}$$

using the method applied in the previous sections in order to generate new exact solutions. For simplicity, we rescale time and space variables so that

$$t = \kappa\tilde{t} \qquad (x, y, z) = \sqrt{D}\,(\tilde{x}, \tilde{y}, \tilde{z})$$

and denote

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu}/\sqrt{D} \qquad \tilde{\kappa} = K/\kappa^2$$

Table 2. Solutions of the equation: $(\mathrm{d}W/\mathrm{d}\xi)^2 = \epsilon(W-a)(W-b)(W-c)$ where $\epsilon = \pm 1$ and a, b, c may be real or complex.

	Solution and corresponding figure		Condition
$\epsilon = -1$	$W = \dfrac{b(a-c) - c(a-b)\,\mathrm{sn}^2(\xi/g;k)}{(a-c)-(a-b)\,\mathrm{sn}^2(\xi/g;k)}$	Figure 2(a)	$c < b < W \leq a$ $g = 2/(a-c)^{1/2}$ $k^2 = (a-b)/(a-c)$
	$W = a - (a-b)\,\mathrm{sn}^2(\xi/g;k)$	Figure 2(b)	$c < b \leq W < a$ $g = 2/(a-c)^{1/2}$
	$k^2 = (a-b)/(a-c)$		
	$W = \dfrac{a\,\mathrm{sn}^2(\xi/g;k) + (c-a)}{\mathrm{sn}^2(\xi/g;k)}$	Figure 2(c)	$W \leq c < b < a$ $g = 2/(a-c)^{1/2}$
	$k^2 = (a-b)/(a-c)$		
	$W = \dfrac{(a+A)\,\mathrm{cn}(\xi/g;k) + (A-a)}{\mathrm{cn}(\xi/g;k) - 1}$	Figure 2(d)	$W \leq a$ $A^2 = (a - \mathrm{Re}\,b)^2 + (\mathrm{Im}\,b)^2$ $g = a^{-1/2}$ $k^2 = (A - \mathrm{Re}\,b + a)/(2A)$
$\epsilon = +1$	$W = c + (b-c)\,\mathrm{sn}^2(\xi/g;k)$	Figure 2(e)	$c < W \leq b < a$ $g = 2/(a-c)^{1/2}$ $k^2 = (b-c)/(a-c)$
	$W = \dfrac{b(a-c) - a(b-c)\,\mathrm{sn}^2(\xi/g;k)}{(a-c)-(b-c)\,\mathrm{sn}^2(\xi/g;k)}$	Figure 2(f)	$c \leq W < b < c$ $g = 2/(a-c)^{1/2}$ $k^2 = (b-c)/(a-c)$
	$W = \dfrac{(a-c) + c\,\mathrm{sn}^2(\xi/g;k)}{\mathrm{sn}^2(\xi/g;k)}$	Figure 2(g)	$c < b < a \leq W$ $g = 2/(a-c)^{1/2}$ $k^2 = (b-c)/(a-c)$
	$W = \dfrac{(A-a)\,\mathrm{cn}(\xi/g;k) + (a+A)}{1 - \mathrm{cn}(\xi/g;k)}$	Figure 2(h)	$W \geq a$ $A^2 = (a - \mathrm{Re}\,b)^2 + (\mathrm{Im}\,b)^2$ $g = a^{-1/2}$ $k^2 = (A + \mathrm{Re}\,b - a)/(2A)$
	$W = (4 + a\xi^2)/\xi^2$	Figure 2(i)	$W \geq a = b = c$
	$W = a + (a-b)\tan[\frac{1}{2}\xi(a-b)^{1/2}]$	Figure 2(j)	$W \geq a > b = c$
	$W = a + (b-a)\tanh[\frac{1}{2}\xi(b-a)]$	Figure 2(k)	$c = b \geq W \geq a$

which transforms equation (4.7) into

$$\mathrm{i}\Phi_{\tilde{t}} + \mathrm{i}\tilde{\boldsymbol{\mu}}\cdot\tilde{\nabla}\Phi + \tilde{\nabla}^2\Phi - \tilde{\kappa}\Phi_{\tilde{t}\tilde{t}} - A_2\Phi - A_4|\Phi|^2\Phi = 0. \qquad (4.8)$$

We first make the following ansatz:

$$\Phi(\tilde{x},\tilde{y},\tilde{z},\tilde{t}) = R(\xi)\exp[\mathrm{i}\theta(\xi) - \mathrm{i}\omega t] \qquad (4.9)$$

where $\xi = \boldsymbol{n}\cdot\tilde{\boldsymbol{x}} - \gamma t$ and $|\boldsymbol{n}|^2 = 1$ and then separate the real and imaginary parts of equation (4.8) to yield

$$-\gamma R' + \tilde{\mu}R' + (2R'\theta' + R\theta'')(1 - \tilde{\kappa}\gamma^2) - 2\omega\hat{\kappa}\gamma R' = 0 \tag{4.10}$$

for the imaginary part and

$$(\gamma\theta' + \omega)R - \tilde{\mu}R\theta' + R'' - R(\theta')^2R'' - \tilde{\kappa}[\gamma^2R'' - R(\gamma\theta' + \omega)^2] - A_2R - A_4R^3 = 0 \tag{4.11}$$

for the real part, with $\tilde{\mu} = \tilde{\boldsymbol{\mu}}\cdot\boldsymbol{n}$ and $' = \mathrm{d}/\mathrm{d}\xi$. The first of the two equations can be integrated once to give

$$\theta'(\xi) = \frac{2\omega\tilde{\kappa}\gamma + \gamma - \tilde{\mu}}{2(1 - \tilde{\kappa}\gamma^2)} + C/R^2 \tag{4.12}$$

where C is an integration constant. Substituting this into equation (4.10) results in an ordinary differential equation in terms of R as

$$\begin{aligned}\tfrac{1}{2}(1 - \tilde{\kappa}\gamma^2)(R')^2 + \tfrac{1}{2}\left\{\omega + \omega^2\tilde{\kappa} - A_2 + \frac{(2\omega\hat{\kappa}\gamma + \gamma - \tilde{\mu})^2}{4(1 - \kappa\gamma^2)}\right\}R^2 + \tfrac{1}{2}C^2(1 - \tilde{\kappa}\gamma^2)R^{-2} \\ - \tfrac{1}{4}A_4R^4 + D = 0.\end{aligned} \tag{4.13}$$

Substituting $W = R^2$ reduces equation (4.13) into a standard form

$$\begin{aligned}\tfrac{1}{8}(1 - \tilde{\kappa}\gamma^2)(W')^2 + \tfrac{1}{2}\left\{\omega + \omega^2\tilde{\kappa} - A_2 + \frac{(2\omega\hat{\kappa}\gamma + \gamma - \tilde{\mu})^2}{4(1 - \kappa\gamma^2)}\right\}W^2 + \tfrac{1}{2}C^2(1 - \tilde{\kappa}\gamma^2) \\ - \tfrac{1}{4}A_4W^3 + DW = 0.\end{aligned} \tag{4.14}$$

Solutions of this equation are tabulated in table 2 and illustrated in figure 2 for the reader's convenience. They take the form of elliptic functions which in limiting cases may become hyperbolic (solitary waves) or trigonometric for $k \to 0$, respectively.

5. Conclusions

In this paper we have examined the equations of motion for the order parameter of many-body systems near criticality which arise in the first order of approximation when the interaction potential has a term linear in momentum. In their recent paper Dixon and Tuszyński (1989) were able to find only rather restrictive form of solutions in this case which assumed that carrier and envelope waves propagate in orthogonal directions. Here, we were able to generalise significantly these results by analysing the original equation of motion in the presence of inelastic scattering processes. The solutions found were obtained by mapping the problem on a derivative nonlinear Schrödinger equation and using both known and new types of solutions. The point of significance is that in $1 + 1$ dimesions this equation is integrable and soliton solutions can always be found. We have also shown reductions in higher dimensions and tabulated the exact solutions of the reduced equations.

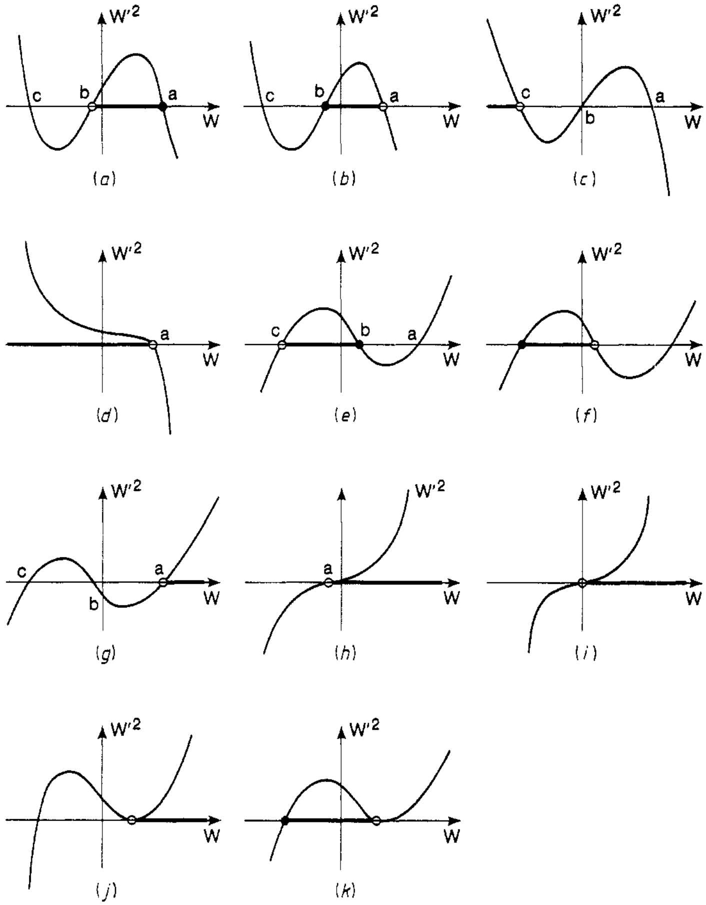

Figure 2. Plots of $(W')^2$ against W for solutions of the equation $(W')^2 = \epsilon(W - a)(W - b)(W - c)$ corresponding to table 2.

A further extension of the model was presented in section 4 where current densities were included resulting from off-diagonal interactions in the microscopic Hamiltonian. In this case, the problem is completely solvable in $1+1$ dimesions and the solutions involve elliptic and hyperbolic functions, including solitary waves.

The main conclusion is that the many-body problem for strongly interacting quantum systems appears to be solvable near criticality in the next order of approximation of the coherent structures method. Dixon and Tuszyński (1989) showed how it maps on the NLS equation in the zeroth order. Here, we have shown that in the first order of approximation the mapping relates it to another integrable equation: the DNLS equation. We intend to pursue the question of solvability of the problem to the second order in the near future.

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